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Derivation of low-temperature expansions for Ising model VI. Three-dimensional lattices—temperature grouping

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Abstract. A brief description is given of the derivation of series expansions for the three-dimensional Ising model of a ferromagnet and antiferromagnet as a low-temperature grouping. New results are given for the ferromagnetic polynomials for the face-centred cubic lattice to order 40; for the ferromagnetic and antiferromagnetic polynomials for the body-centred cubic lattice to order 28, for the simple cubic lattice to order 20 and for the diamond lattice to order 15.

1. Introduction and summary

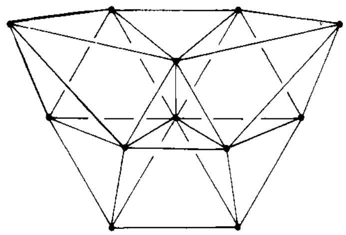
In this paper we extend the series expansions of four three-dimensional lattices: the face-centred cubic, body-centred cubic, simple cubic and diamond, as a temperature or *u grouping*. We have introduced the problem and defined the notation in previous papers (Sykes *et al* 1965, 1973a, b, c to be referred to as I, II, III, IV respectively); data for the field or μ grouping for three-dimensional lattices are given in I and Sykes *et al* 1973d (to be referred to as V).

We give new results for the ferromagnetic polynomials through ψ_{40} for the face-centred cubic lattice; for the ferromagnetic polynomials through ψ_{28} and the corresponding antiferromagnetic polynomials ψ^{a}_{24} through ψ^{a}_{28} for the body-centred cubic lattice; through ψ_{20} and ψ^{a}_{19}, ψ^{a}_{20} for the simple cubic lattice; through ψ_{15} and ψ^{a}_{13}, ψ^{a}_{14}, ψ^{a}_{15} for the diamond lattice. From these polynomials we derive zero-field expansions to the same order for the configurational free energy, the spontaneous magnetization, and the ferromagnetic and antiferromagnetic susceptibilities.

The expansions have application to the elucidation of the physical properties of the model and are of special interest in the theory of scaling. We make an analysis of some of the new data in a companion paper (Gaunt and Sykes 1973).

2. Face-centred cubic and diamond lattices

For the face-centred cubic lattice the complete high-field polynomials L_1–L_8 derived in I and V provide on re-arrangement the u grouping through u^{32}; to extend the expansion through u^{40} it is necessary to include contributions from configurations with up to 12 spins. The leading term of ψ_{40} corresponds to an unique configuration with 12 spins and 32 bonds:



24N

which contributes $24\mu^{12}$. The next term, $5298\mu^{11}$, corresponds to numerous configurations of 11 spins and 26 bonds. We have completed the polynomials through ψ_{40} and list them in appendix 1.

To obtain the u grouping for the diamond lattice we have derived partial codes for the diamond–face-centred cubic system by a straightforward extension of the theory of IV, § 2. Each code $(\lambda, \alpha, \beta, \gamma, \delta)$ can be interpreted in two ways; on the face-centred cubic lattice by the substitution (3.9) of II which reduces in zero field to

$$(\lambda, \alpha, \beta, \gamma, \delta) = u^{\frac{1}{2}(3\alpha+4\beta+3\gamma)} \tag{2.1}$$

and on the diamond lattice by the substitutions (2.6) and (2.7) of II which reduce in zero field to

$$(\lambda, \alpha, \beta, \gamma, \delta) = u^{\frac{1}{2}(\alpha+2\beta+\gamma)}\left(\frac{1+u}{1+u^2}\right)^{\alpha+\gamma}\left(\frac{2}{1+u^2}\right)^{\beta}. \tag{2.2}$$

It follows from a comparison of (2.1) and (2.2) that to obtain all the contributions to a given power of u on the diamond lattice it would suffice to encode all configurations that contribute to the cube of that power, or less, on the face-centred cubic lattice. However this condition is only necessary for codes with $\beta = 0$ and in practice we have been able to obtain the partial codes to complete ψ_{15} on the diamond lattice by encoding all the configurations on the face-centred cubic used to derive the u grouping through u^{40} together with a few extra configurations found by inspection. We give the ferromagnetic polynomials in appendix 1 and the corresponding antiferromagnetic polynomials in appendix 2. These latter are defined by (1.11) of II:

$$\ln \Lambda^{\mathrm{a}} = \sum_s \psi_s^{\mathrm{a}}(\theta) w^s. \tag{2.3}$$

These two sets of polynomials are the most generally useful; they determine the susceptibilities through the relations

$$\chi(u) = \left(2\mu\frac{\partial}{\partial\mu}\right)^2 L\bigg|_{\mu=1} \qquad L = \ln\Lambda \tag{2.4}$$

$$\chi^{\mathrm{a}}(w) = \left(2\mu\frac{\partial}{\partial\mu}\right)^2 L^{\mathrm{a}}\bigg|_{\mu=1} \qquad L^{\mathrm{a}} = \ln\Lambda^{\mathrm{a}}. \tag{2.5}$$

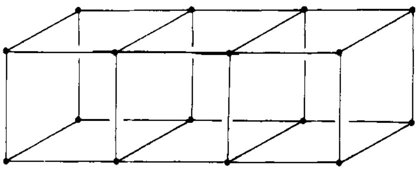
The partial codes are numerous and can be used to provide the more detailed information required for the *ferrimagnetic* polynomials (1.7) of II:

$$\ln \Lambda = \sum_s \psi_s(\mu, v) u^s. \tag{2.6}$$

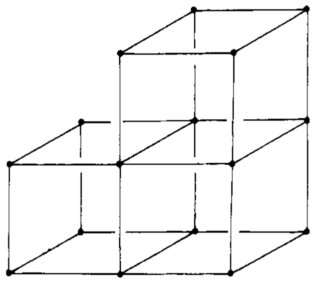
The enumerations required to provide the data in appendixes 1 and 2 are extensive; considerable care has been taken to check the derivation and it is hoped that errors have been eliminated.

3. Simple cubic and body-centred cubic lattices

We have extended the u grouping for the simple cubic lattice by direct enumeration through u^{20}. The ferromagnetic polynomials are listed in appendix 1. The leading term of ψ_{20} corresponds to two configurations of 16 spins and 28 bonds:



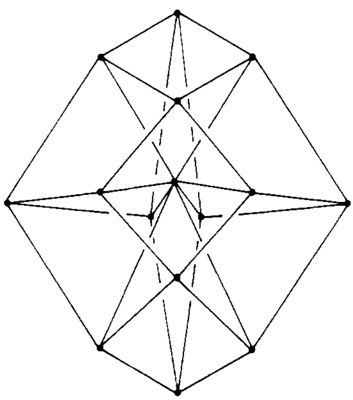
$3N$



$12N$

In contrast to the shadow system of octahedra (V, § 3), which is visually difficult, direct enumeration is visually simple and straightforward although tedious. We have derived the corresponding partial codes and the antiferromagnetic polynomials are listed in appendix 2.

For the body-centred cubic lattice the leading term of ψ_{28} corresponds to a configuration of 15 spins and 32 bonds:



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which may be described as the 'fourteen-neighbour figure' obtained by taking a site and its eight first and six second neighbours.

The sub-lattice division is 7–8 and the corresponding seventh order code, (32, 24, 0, 0, 8) results from a cube with six other cubes, one on each face. In contradistinction to the simple cubic lattice the shadow system (V, § 3) is visually simple but direct enumeration awkward as the configurations are difficult to classify. We have extended the u grouping on the body-centred cubic lattice through u^{28} by deriving the partial codes; the ferromagnetic and antiferromagnetic polynomials are listed in the appendixes. A method of deriving codes for the body-centred cubic lattice by electronic computer is described by Elliott (1969) who gives some higher terms.

4. Expansions for zero magnetic field

From the results of §§ 2 and 3 and appendix 1 the corresponding series expansions for the physical properties in zero magnetic field are readily derived. For the face-centred cubic lattice we obtain twelve new coefficients for the reduced configurational free energy and spontaneous magnetization, to supplement the expansions given in I, appendix 4:

$$\begin{aligned}\ln \Lambda(u) = {} & \ldots - 41526u^{29} + 8777\tfrac{1}{5}u^{30} - 61446u^{31} - 54402u^{32} + 772624u^{33} \\ & - 1317960u^{34} + 661848u^{35} - 820665\tfrac{1}{6}u^{36} + 1549408u^{37} \\ & + 8084382u^{38} - 28589452u^{39} + 29889394\tfrac{1}{2}u^{40} \ldots \end{aligned} \tag{4.1}$$

$$\begin{aligned}I(u) = {} & \ldots + 406056u^{29} - 79532u^{30} + 729912u^{31} + 631608u^{32} - 9279376u^{33} \\ & + 15771600u^{34} - 7467336u^{35} + 10935114u^{36} - 21835524u^{37} \\ & - 112752684u^{38} + 400576168u^{39} - 410287368u^{40} \ldots . \end{aligned} \tag{4.2}$$

From (4.1) the reduced configurational energy U and the specific heat at constant field C_H follow through the defining relations

$$U(u) = 4u\frac{\partial L}{\partial u} \tag{4.3}$$

$$\frac{C_H}{R(\ln u)^2} = \frac{1}{4}u\frac{\partial U}{\partial u}. \tag{4.4}$$

The ferromagnetic susceptibility is not given in I and we quote the expansion in full:

$$\begin{aligned}\tfrac{1}{4}\chi(u) = {} & u^6 + 24u^{11} - 26u^{12} + 72u^{15} + 378u^{16} - 1080u^{17} + 665u^{18} + 384u^{19} \\ & + 1968u^{20} + 2016u^{21} - 25698u^{22} + 39552u^{23} - 3872u^{24} \\ & + 20880u^{25} - 65727u^{26} - 379072u^{27} + 1277646u^{28} - 986856u^{29} \\ & + 176978u^{30} - 2163504u^{31} - 1818996u^{32} + 27871080u^{33} \\ & - 47138844u^{34} + 20789424u^{35} - 36509652u^{36} + 77055330u^{37} \\ & + 393046656u^{38} - 1402934816u^{39} + 1403843388u^{40} \ldots . \end{aligned} \tag{4.5}$$

For the diamond lattice we obtain three new coefficients for the configurational free energy and spontaneous magnetization to supplement the expansions given by Essam and Sykes (1963) and in I, appendix 4, together with the ferromagnetic susceptibility through u^{15}:

$$\ln \Lambda(u) = \ldots + 26956u^{13} + 93140\tfrac{1}{7}u^{14} + 329258\tfrac{2}{5}u^{15} \ldots \tag{4.6}$$

$$I(u) = \ldots - 2924680u^{13} - 11596284u^{14} - 46364456u^{15} \ldots \tag{4.7}$$

$$\begin{aligned}\tfrac{1}{4}\chi(u) = u^2 &+ 8u^3 + 44u^4 + 208u^5 + 984u^6 + 4584u^7 + 21314u^8 + 98292u^9 \\ &+ 448850u^{10} + 2038968u^{11} + 9220346u^{12} + 41545564u^{13} \\ &+ 186796388u^{14} + 838623100u^{15} + \ldots .\end{aligned} \tag{4.8}$$

From the data in appendix 2 we obtain three new coefficients for the antiferromagnetic susceptibility to supplement appendix 5 of I:

$$\chi^{\mathrm{a}}(y) = \ldots + 231728y^{26} + 863664y^{28} + 3313392y^{30} \ldots . \tag{4.9}$$

For the simple cubic lattice we obtain two new coefficients for the configurational free energy, spontaneous magnetization and antiferromagnetic susceptibility to supplement appendixes 4 and 5 of I, together with the ferromagnetic susceptibility through u^{20}:

$$\ln \Lambda(u) = \ldots + 6583341u^{19} - 20852363\tfrac{1}{4}u^{20} \ldots \tag{4.10}$$

$$I(u) = \ldots - 101585544u^{19} + 338095596u^{20} \ldots \tag{4.11}$$

$$\chi^{\mathrm{a}}(y) = \ldots + 53457696y^{38} - 177637248y^{40} \ldots \tag{4.12}$$

$$\begin{aligned}\tfrac{1}{4}\chi(u) = u^3 &+ 12u^5 - 14u^6 + 135u^7 - 276u^8 + 1520u^9 - 4056u^{10} + 17778u^{11} \\ &- 54392u^{12} + 213522u^{13} - 700362u^{14} + 2601674u^{15} - 8836812u^{16} \\ &+ 31925046u^{17} - 110323056u^{18} + 393008712u^{19} - 1369533048u^{20} \ldots .\end{aligned} \tag{4.13}$$

For the body-centred cubic lattice the corresponding expansions are extended by five coefficients and the ferromagnetic susceptibility is obtained through u^{28}:

$$\begin{aligned}\ln \Lambda(u) = \ldots &+ 3832961\tfrac{1}{2}u^{24} - 7941796u^{25} + 1118118u^{26} \\ &+ 43016052u^{27} - 133595088\tfrac{6}{7}u^{28}\end{aligned} \tag{4.14}$$

$$\begin{aligned}I(u) = \ldots &- 54012882u^{24} + 112640896u^{25} - 5164464u^{26} \\ &- 694845120u^{27} + 2160781086u^{28} \ldots\end{aligned} \tag{4.15}$$

$$\begin{aligned}\chi^{\mathrm{a}}(y) = \ldots &+ 29452776y^{48} - 61952896y^{50} + 4795392y^{52} \\ &+ 374024448y^{54} - 1173895476y^{56} \ldots\end{aligned} \tag{4.16}$$

$$\begin{aligned}\tfrac{1}{4}\chi(u) = u^4 &+ 16u^7 - 18u^8 + 252u^{10} - 576u^{11} + 519u^{12} + 3264u^{13} - 12468u^{14} \\ &+ 20568u^{15} + 26662u^{16} - 215568u^{17} + 528576u^{18} - 164616u^{19} \\ &- 3014889u^{20} + 10894920u^{21} - 13796840u^{22} - 29909616u^{23} \\ &+ 190423962u^{24} - 399739840u^{25} - 22768752u^{26} + 2803402560u^{27} \\ &- 8743064909u^{28} \ldots .\end{aligned} \tag{4.17}$$

Acknowledgments

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Appendix 1. Ferromagnetic polynomials

Face-centred cubic lattice

$$\psi_6 = \mu$$
$$\psi_{11} = 6\mu^2$$
$$\psi_{12} = -6\tfrac{1}{2}\mu^2$$
$$\psi_{15} = 8\mu^3$$
$$\psi_{16} = 42\mu^3$$
$$\psi_{17} = -120\mu^3$$
$$\psi_{18} = 2\mu^4+70\tfrac{1}{3}\mu^3$$
$$\psi_{19} = 24\mu^4$$
$$\psi_{20} = 123\mu^4$$
$$\psi_{21} = 126\mu^4$$
$$\psi_{22} = 30\mu^5-1653\mu^4$$
$$\psi_{23} = 96\mu^5+2322\mu^4$$
$$\psi_{24} = \mu^6+448\mu^5-944\tfrac{1}{4}\mu^4$$
$$\psi_{25} = 30\mu^6+792\mu^5$$
$$\psi_{26} = 168\mu^6-2871\mu^5$$
$$\psi_{27} = 8\mu^7+776\mu^6-16296\mu^5$$
$$\psi_{28} = 36\mu^7+1212\mu^6+49290\mu^5$$
$$\psi_{29} = 336\mu^7+3930\mu^6-45792\mu^5$$
$$\psi_{30} = 28\mu^8+1350\mu^7-6904\mu^6+14303\tfrac{1}{5}\mu^5$$
$$\psi_{31} = 96\mu^8+3528\mu^7-65070\mu^6$$
$$\psi_{32} = 786\mu^8+9036\mu^7-64224\mu^6$$
$$\psi_{33} = 80\mu^9+2432\mu^8-1160\mu^7+771272\mu^6$$
$$\psi_{34} = 438\mu^9+9804\mu^8+1038\mu^7-1329240\mu^6$$
$$\psi_{35} = 6\mu^{10}+1776\mu^9+19314\mu^8-281400\mu^7+922152\mu^6$$
$$\psi_{36} = 270\mu^{10}+6520\mu^9+29146\mu^8-622498\mu^7-234103\tfrac{1}{6}\mu^6$$
$$\psi_{37} = 1464\mu^{10}+23482\mu^9+20550\mu^8+1503912\mu^7$$
$$\psi_{38} = 96\mu^{11}+5844\mu^{10}+45351\mu^9-322950\mu^8+8356041\mu^7$$

$$\psi_{39} = 848\mu^{11} + 19154\mu^{10} + 126016\mu^{9} - 474806\mu^{8} - 28260664\mu^{7}$$

$$\psi_{40} = 24\mu^{12} + 5298\mu^{11} + 54066\mu^{10} + 52884\mu^{9} - 4371355\tfrac{1}{2}\mu^{8} + 34148478\mu^{7}.$$

Diamond lattice

$$\psi_{2} = \mu$$

$$\psi_{3} = 2\mu^{2}$$

$$\psi_{4} = 6\mu^{3} - 2\tfrac{1}{2}\mu^{2}$$

$$\psi_{5} = 22\mu^{4} - 16\mu^{3}$$

$$\psi_{6} = 2\mu^{6} + 91\mu^{5} - 91\mu^{4} + 10\tfrac{1}{3}\mu^{3}$$

$$\psi_{7} = 24\mu^{7} + 396\mu^{6} - 512\mu^{5} + 122\mu^{4}$$

$$\psi_{8} = \mu^{10} + 6\mu^{9} + 207\mu^{8} + 1746\mu^{7} - 2877\mu^{6} + 1054\mu^{5} - 53\tfrac{1}{4}\mu^{4}$$

$$\psi_{9} = 16\mu^{11} + 102\mu^{10} + 1508\mu^{9} + 7574\mu^{8} - 16072\mu^{7} + 8066\tfrac{2}{3}\mu^{6} - 944\mu^{5}$$

$$\begin{aligned}\psi_{10} = {} & 2\mu^{14} + 12\mu^{13} + 198\mu^{12} + 1120\mu^{11} + 9834\mu^{10} + 31365\mu^{9} - 88765\mu^{8} \\ & + 57749\mu^{7} - 11058\mu^{6} + 311\tfrac{1}{5}\mu^{5}\end{aligned}$$

$$\begin{aligned}\psi_{11} = {} & 40\mu^{15} + 240\mu^{14} + 2064\mu^{13} + 9894\mu^{12} + 58920\mu^{11} + 118568\mu^{10} \\ & - 482136\mu^{9} + 395018\mu^{8} - 107608\mu^{7} + 7442\mu^{6}\end{aligned}$$

$$\begin{aligned}\psi_{12} = {} & 6\mu^{18} + 42\mu^{17} + 626\mu^{16} + 3166\mu^{15} + 18836\mu^{14} + 75536\mu^{13} + 327231\mu^{12} \\ & + 368354\mu^{11} - 2562436\tfrac{1}{2}\mu^{10} + 2607402\mu^{9} - 939367\tfrac{1}{2}\mu^{8} \\ & + 110586\mu^{7} - 1971\tfrac{5}{6}\mu^{6}\end{aligned}$$

$$\begin{aligned}\psi_{13} = {} & 144\mu^{19} + 990\mu^{18} + 7920\mu^{17} + 33634\mu^{16} + 154248\mu^{15} + 515810\mu^{14} \\ & + 1682888\mu^{13} + 558252\mu^{12} - 13251196\mu^{11} + 16692444\mu^{10} \\ & - 7618128\mu^{9} + 1309590\mu^{8} - 59640\mu^{7}\end{aligned}$$

$$\begin{aligned}\psi_{14} = {} & 22\mu^{22} + 192\mu^{21} + 2796\mu^{20} + 15501\mu^{19} + 85696\mu^{18} + 308249\mu^{17} \\ & + 1148682\mu^{16} + 3199836\mu^{15} + 7916204\mu^{14} - 4273104\mu^{13} \\ & - 66199876\mu^{12} + 103918606\mu^{11} - 58481770\mu^{10} \\ & + 13515382\mu^{9} - 1076491\mu^{8} + 13215\tfrac{1}{7}\mu^{7}\end{aligned}$$

$$\begin{aligned}\psi_{15} = {} & 2\mu^{26} + 12\mu^{25} + 54\mu^{24} + 788\mu^{23} + 5610\mu^{22} + 42796\mu^{21} + 191978\mu^{20} \\ & + 816892\mu^{19} + 2517586\mu^{18} + 7844420\mu^{17} + 18106680\mu^{16} \\ & + 32944408\mu^{15} - 58478578\mu^{14} - 316157712\mu^{13} + 629834815\tfrac{1}{3}\mu^{12} \\ & - 429703296\mu^{11} + 126923238\tfrac{2}{5}\mu^{10} - 15044957\tfrac{1}{3}\mu^{9} + 484522\mu^{8}.\end{aligned}$$

Simple cubic lattice

$$\psi_{3} = \mu$$

$$\psi_{5} = 3\mu^{2}$$

$$\psi_{6} = -3\tfrac{1}{2}\mu^{2}$$

$$\psi_7 = 15\mu^3$$
$$\psi_8 = 3\mu^4 - 36\mu^3$$
$$\psi_9 = 83\mu^4 + 21\tfrac{1}{3}\mu^3$$
$$\psi_{10} = 48\mu^5 - 328\tfrac{1}{2}\mu^4$$
$$\psi_{11} = 18\mu^6 + 426\mu^5 + 405\mu^4$$
$$\psi_{12} = \mu^8 + 8\mu^7 + 496\mu^6 - 2804\mu^5 - 162\tfrac{3}{4}\mu^4$$
$$\psi_{13} = 378\mu^7 + 1575\mu^6 + 5532\mu^5$$
$$\psi_{14} = 24\mu^9 + 306\mu^8 + 3888\mu^7 - 22144\tfrac{1}{2}\mu^6 - 4608\mu^5$$
$$\psi_{15} = 24\mu^{10} + 127\mu^9 + 4622\mu^8 - 1360\mu^7 + 64574\mu^6 + 1406\tfrac{1}{5}\mu^5$$
$$\psi_{16} = 3\mu^{12} + 24\mu^{11} + 396\mu^{10} + 5544\mu^9 + 22396\tfrac{1}{2}\mu^8 - 157380\mu^7 - 84738\mu^6$$
$$\psi_{17} = 660\mu^{11} + 4131\mu^{10} + 40050\mu^9 - 106113\mu^8 + 674652\mu^7 + 53370\mu^6$$
$$\psi_{18} = 96\mu^{13} + 1080\mu^{12} + 6656\mu^{11} + 67267\mu^{10} + 60804\mu^9 - 947582\tfrac{1}{2}\mu^8 - 1261904\mu^7 - 13150\tfrac{2}{3}\mu^6$$
$$\psi_{19} = 114\mu^{14} + 732\mu^{13} + 11562\mu^{12} + 70275\mu^{11} + 236808\mu^{10} - 1368954\mu^9 + 6392769\mu^8 + 1240035\mu^7$$
$$\psi_{20} = 15\mu^{16} + 168\mu^{15} + 2340\mu^{14} + 23976\mu^{13} + 101685\mu^{12} + 602928\mu^{11} - 614784\mu^{10} - 3978300\mu^9 - 16362155\tfrac{1}{4}\mu^8 - 628236\mu^7.$$

Body-centred cubic lattice

$$\psi_4 = \mu$$
$$\psi_7 = 4\mu^2$$
$$\psi_8 = -4\tfrac{1}{2}\mu^2$$
$$\psi_{10} = 28\mu^3$$
$$\psi_{11} = -64\mu^3$$
$$\psi_{12} = 12\mu^4 + 36\tfrac{1}{3}\mu^3$$
$$\psi_{13} = 204\mu^4$$
$$\psi_{14} = 12\mu^5 - 798\mu^4$$
$$\psi_{15} = 216\mu^5 + 948\mu^4$$
$$\psi_{16} = 27\mu^6 + 1262\mu^5 - 366\tfrac{1}{4}\mu^4$$
$$\psi_{17} = 312\mu^6 - 9072\mu^5$$
$$\psi_{18} = 72\mu^7 + 2368\mu^6 + 17592\mu^5$$
$$\psi_{19} = 4\mu^8 + 704\mu^7 + 4312\mu^6 - 14184\mu^5$$
$$\psi_{20} = 198\mu^8 + 4404\mu^7 - 92992\mu^6 + 4174\tfrac{1}{5}\mu^5$$
$$\psi_{21} = 24\mu^9 + 2016\mu^8 + 17616\mu^7 + 275021\tfrac{1}{3}\mu^6$$

$$\psi_{22} = 692\mu^9+10300\mu^8-36348\mu^7-353640\mu^6$$

$$\psi_{23} = 156\mu^{10}+5816\mu^9+41352\mu^8-833064\mu^7+216036\mu^6$$

$$\psi_{24} = 12\mu^{11}+2418\mu^{10}+30714\mu^9+55536\mu^8+3795726\mu^7-51444\tfrac{1}{2}\mu^6$$

$$\psi_{25} = 800\mu^{11}+19568\mu^{10}+99648\mu^9-989076\mu^8-7072736\mu^7$$

$$\psi_{26} = 168\mu^{12}+9720\mu^{11}+89832\mu^{10}+226692\mu^9-6007194\mu^8+6798900\mu^7$$

$$\psi_{27} = 24\mu^{13}+3924\mu^{12}+65112\mu^{11}+312984\mu^{10}-887688\mu^9+46866408\mu^8-3344712\mu^7$$

$$\psi_{28} = \mu^{15}+14\mu^{14}+1327\mu^{13}+39762\mu^{12}+302497\mu^{11}+534960\mu^{10} -13103579\mu^9-122039509\mu^8+669438\tfrac{1}{7}\mu^7.$$

Appendix 2. Antiferromagnetic polynomials†

Diamond lattice

$$2\psi^{a}_{13} = 19847640-16137864\theta_1+8621096\theta_2-2892028\theta_3+564124\theta_4 -53872\theta_5+1680\theta_6$$

$$2\psi^{a}_{14} = -117002370+96624135\theta_1-53722476\theta_2+19545831\theta_3 -4354202\theta_4+527656\theta_5-26874\theta_6+255\tfrac{1}{7}\theta_7$$

$$2\psi^{a}_{15} = 699197220\tfrac{4}{5}-583448888\theta_1+336957492\theta_2-131033165\tfrac{1}{3}\theta_3+32688680\theta_4 -4766732\theta_5+341125\tfrac{1}{3}\theta_6-7864\theta_7.$$

Simple cubic lattice

$$2\psi^{a}_{19} = 7244418-24948\theta_1+2883306\theta_2-31620\theta_3+135612\theta_4-1344\theta_5+126\theta_6$$

$$2\psi^{a}_{20} = -18000400\tfrac{1}{2}-3488844\theta_1-7461546\theta_2-486444\theta_3-410481\theta_4 -4176\theta_5-672\theta_6.$$

Body-centred cubic lattice

$$2\psi^{a}_{24} = 13637\tfrac{1}{3}+3388704\theta_1+51\theta_2+435228\theta_3-360\theta_4+2520\theta_5-\tfrac{1}{6}\theta_6$$

$$2\psi^{a}_{25} = -1171896-6119824\theta_1-378784\theta_2-845912\theta_3-4784\theta_4-6552\theta_5+8\theta_6$$

$$2\psi^{a}_{26} = -6137236+6109952\theta_1-2683100\theta_2+916288\theta_3-165252\theta_4 +9072\theta_5-224\theta_6$$

$$2\psi^{a}_{27} = 50168184-3575352\theta_1+20925912\theta_2-585144\theta_3+1171296\theta_4 -6768\theta_5+2016\theta_6$$

$$2\psi^{a}_{28} = -127744158-10745037\theta_1-54289980\theta_2-1379391\theta_3-3294524\theta_4 -5889\theta_5-8190\theta_6+1\tfrac{1}{7}\theta_7.$$

† This supplements appendix 5 of I. For consistency we quote the results for $2\ln\Lambda^{a}$.

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